MTH 295
Fall 2019
Homework 4
Due Thursday, 10/3

Name: Key

1) An equation of the form $y'+p(x)y=q(x)y^n$ is a Bernoulli equation. In class we did the specific example where $n=2$.

a) Show that the substitution $v=y^{1-n}$ always transforms a Bernoulli equation into a linear equation in v if $n\neq 0,1$.

If $v=y^{1-n}$

then $\frac{dv}{dx}=(1-n)y^{-n}\frac{dy}{dx}=(1-n)\frac{1}{y^n}\frac{dy}{dx}$

Divide original ODE by y^n –

$\frac{1}{y^n}y'+p(x)\cdot y^{1-n}=g(x)$

$\frac{1}{(1-n)}\frac{dv}{dx}+p(x)v=g(x)$

or $\frac{dv}{dx}+(1-n)p(x)v=(1-n)g(x)$ which is linear in v.

b) Find the general solution of $x^2y'+2xy-y^3=0$ for $x>0$. You can express y as two explicit functions of x.

$x^2y'+2xy=y^3$

$y'+\frac{2}{x}y=\frac{1}{x^2}y^3$ is Bernoulli, $n=3$

let $v=y^{-2}$

and $\frac{dv}{dx}=-2\frac{1}{y^3}\cdot\frac{dy}{dx}$

Divide original ODE by y^3 –

$\frac{1}{y^3}y'+\frac{2}{x}\cdot\frac{1}{y^2}=\frac{1}{x^2}$

$-\frac{1}{2}\frac{dv}{dx}+\frac{2}{x}v=\frac{1}{x^2}$

$\frac{dv}{dx}-\frac{4}{x}v=-\frac{2}{x^2}$ is linear so there exists an integrating factor $\mu=e^{-\int\frac{4}{x}dx}=e^{-4\ln|x|}=\frac{1}{x^4}$

Then $v(x)=\frac{1}{\mu}\int\frac{1}{x^4}\left(-\frac{2}{x^2}\right)dx=x^4\left\{\frac{2}{5x^5}+c\right\}=\frac{2}{5x}+cx^4$

so $y^2=\frac{1}{\frac{2}{5x}+cx^4}=\frac{5x}{2+cx^5}$ and $\boxed{y=\pm\sqrt{\frac{5x}{2+cx^5}}}$ which you should check!

2) An equation of the form $y = F(p)$ where $p = \frac{dy}{dx}$ can, in theory, be solved. Solve the equation $y = xy' + \frac{1}{y'}$. First, differentiate the given equation with respect to x. The result will be an algebraic equation in p and p'. Find the roots of this algebraic equation and solve for y. You will actually find two distinct sets of solutions, one of which will be an infinite set of lines. The other solution is called a "singular solution" because it will not contain an unknown constant, i.e. it is one solution, not an infinite set.

$$y = xy' + \frac{1}{y'}$$

so $$y' = y' + xy'' - \frac{1}{(y')^2} y''$$

$$0 = xy'' - \frac{1}{(y')^2} y''$$

but $y' = p$, $y'' = p'$.

so $$0 = xp' - \frac{1}{p^2} p'$$

which factors –

$$p'\left(x - \frac{1}{p^2}\right) = 0$$

so $p' = 0$ or $x - \frac{1}{p^2} = 0$

if $p' = 0$ then $p = c$ and substituting –

$$y = xp + \frac{1}{p} = cx + \frac{1}{c}$$

if $x - \frac{1}{p^2} = 0$ then

$$p = y' = \pm \frac{1}{\sqrt{x}}$$

and substituting –

$$y = x\left(\pm \frac{1}{\sqrt{x}}\right) \pm \sqrt{x}$$

$$= \pm 2\sqrt{x}$$

or $y^2 = 4x$

So there are 2 solutions,

$$\boxed{\begin{array}{l} y^2 = 4x \\ y = cx + \frac{1}{c} \end{array}}$$

* note; some of you said –

if $y' = \pm \frac{1}{\sqrt{x}}$ then $y = \pm 2\sqrt{x} + c$ by integrating, but if you substitute into original equation you find

$$\pm 2\sqrt{x} + c = x\left(\pm \frac{1}{\sqrt{x}}\right) + (\pm \sqrt{x})$$

$$= \pm 2\sqrt{x}$$

So it must be true that $c = 0$ and $y = \pm 2\sqrt{x}$. This is the singular solution which results from the non-linearity of the original ODE.

3) Find the general solution of the homogeneous equation $\frac{dy}{dx} = -\frac{4x+3y}{2x+y}$.

$$\frac{dy}{dx} = -\frac{4+3(y/x)}{2+(y/x)}$$

let $v = y/x$, $y = xv$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = -\frac{4+3v}{2+v}$$

$$x\frac{dv}{dx} = \frac{-4-3v-2v-v^2}{2+v}$$

$$= -\frac{v^2+5v+4}{2+v}$$ which is separable -

$$\frac{2+v}{(v+1)(v+4)}\,dv = -\frac{dx}{x}$$

$$\frac{2+v}{(v+1)(v+4)} = \frac{A}{v+1} + \frac{B}{v+4}$$

$2+v = A(v+4) + B(v+1)$

$-2 = -3B$

$B = 2/3$

$1 = 3A$

$A = \frac{1}{3}$

$$\int\left\{\frac{1}{3}\cdot\frac{1}{v+1} + \frac{2}{3}\cdot\frac{1}{v+4}\right\}dv = -\ln|x| + c$$

$$\frac{1}{3}\ln|v+1| + \frac{2}{3}\ln|v+4| = -\ln|x| + c$$

$$\ln|v+1| + \ln(v+4)^2 = -\ln|x^3| + c$$

$$\ln\{|v+1|(v+4)^2\} = \ln\left|\frac{1}{x^3}\right| + c$$

$$|v+1|(v+4)^2 = c\left|\frac{1}{x^3}\right|$$

$$|x^3|\left|\frac{y}{x}+1\right|\left(\frac{y}{x}+4\right)^2 = c$$

$$\boxed{|y+x|(y+4x)^2 = c}$$

4) a) Show that $f(x,y)=\dfrac{2xye^{(x/y)^2}}{y^2+y^2e^{(x/y)^2}+2x^2e^{(x/y)^2}}$ can be written in the form $F(x/y)$ and thus the equation $y'=f(x,y)$ is homogeneous.

$$f(x,y)=\frac{2\left(\frac{x}{y}\right)e^{(x/y)^2}}{1+e^{(x/y)^2}+2\left(\frac{x}{y}\right)^2e^{(x/y)^2}}=F(x,y)$$

b) Find the general solution of the first order ODE $y'(x)=f(x,y)$ by making the substitution $v=x/y$. Express your solution in the form $y=y(v)$. In other words, express y as a function of x/y.

from above – $$\frac{dx}{dy}=\frac{1+e^{(x/y)^2}+2\left(\frac{x}{y}\right)^2e^{(x/y)^2}}{2\left(\frac{x}{y}\right)e^{(x/y)^2}}$$

Let $v=\frac{x}{y}$. Then $x=yv$, and $\frac{dx}{dy}=v+y\frac{dv}{dy}$

$$v+y\frac{dv}{dy}=\frac{1+e^{v^2}+2v^2e^{v^2}}{2ve^{v^2}}$$

$$y\frac{dv}{dy}=\frac{1+e^{v^2}}{2ve^{v^2}}$$

$$\frac{2ve^{v^2}dv}{1+e^{v^2}}dv=\frac{dy}{y}$$

$$\ln(1+e^{v^2})=\ln|y|+c$$

$$1+e^{v^2}=c|y|$$

$$|y|=c(1+e^{(x/y)^2})$$

or $\boxed{y=c(1+e^{(x/y)^2})}$ since c may be positive or negative.

5) Find the general solution of the Riccati equation $y' = -\frac{1}{x^2} - \frac{y}{x} + y^2$ given that $y_1(x) = \frac{1}{x}$ is a particular solution. Your solution will be explicit.

We showed in class that if

$$\frac{dy}{dx} = q_1(x) + q_2(x)y + q_3(x)y^2, \quad y_1(x) \text{ is any particular solution,}$$

and we assume $y(x) = y_1(x) + \frac{1}{v(x)}$, then v satisfies

$$\frac{dv}{dx} = -(q_2 + 2q_3 y_1)v - q_3$$

In this case, $y_1 = \frac{1}{x}$, $q_2 = -\frac{1}{x}$, $q_3 = 1$

so $\frac{dv}{dx} = -\left(-\frac{1}{x} + \frac{2}{x}\right)v - 1$

$$\frac{dv}{dx} = -\frac{1}{x}v - 1$$

$$\frac{dv}{dx} + \frac{1}{x}v = -1$$

This is linear so there exists an integrating factor

$$\mu = e^{\int \frac{dx}{x}} = x$$

and $v = \frac{1}{x}\int x(-1)\,dx = \frac{1}{x}\left(-\frac{x^2}{2} + c\right) = \frac{1}{2x}(c - x^2)$

and $y = \frac{1}{x} + \frac{1}{v}$

$$\boxed{y = \frac{1}{x} + \frac{2x}{c - x^2}}$$